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Generalization of Blocks for *D*-Lattices and Lattice-Ordered Effect Algebras

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We show that every *D*-lattice (lattice-ordered effect algebra) P is a set-theoretic union of maximal subsets of mutually compatible elements, called blocks. Moreover, blocks are sub-*D*-lattices and sub-effect-algebras of P which are *MV*-algebras closed with respect to all suprema and infima existing in P.

1. INTRODUCTION AND BASIC DEFINITIONS

Kôpka [11] introduced a new algebraic structure of fuzzy sets, a *D*-poset of fuzzy sets. A difference of comparable fuzzy sets is a primary operation in this structure. Later, Kôpka and Chovanec [13], by transferring the properties of a difference operation of *D*-poset of fuzzy sets to an arbitrary partially ordered set, obtained a new algebraic structure, a *D*-poset that generalizes orthoalgebras and MV-algebras (see also ref. 6).

Definition 1.1. [13] Let $(P, \leq)$ be a poset with the least element 0 and the greatest element 1. Let $\ominus$ be a partial binary operation on P such that $b \ominus a$ is defined iff $a \leq b$. Then $(P; \leq, \ominus, 0, 1)$ is called a *difference poset* (*D-poset*) if the following conditions are satisfied:

(Di) For any $a \in P$, $a \ominus 0 = a$.
(Dii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Effect algebras (introduced by Foulis and Bennett [5]) are important for

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modeling unsharp measurements in Hilbert space: The set of all effects is the set of all self-adjoint operators T on a Hilbert space H with $0 \leq T \leq 1$. In a general algebraic form an effect algebra is defined as follows:

Definition 1.2. A structure $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on P which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus$ is defined.
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined.
(Eiii) For every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$.
(Eiv) If $1 \oplus a$ is defined, then $a = 0$.

Later Foulis proved that, if, on a D-poset $(P; \leq, \ominus, 0, 1)$ [effect algebra $(P; \oplus, 0, 1)$] a partial operation $\oplus$ $(\ominus)$ is defined by

$$a \oplus b \text{ is defined and } a \oplus b = c \text{ iff } a \leq c \text{ and } c \ominus a = b$$

then P becomes an effect algebra (D-poset). On the other hand in ref. 18 it has been shown that if Q is a subalgebra of an effect algebra $(E; \oplus, 0, 1)$ (i.e., $1 \in Q$ and $a, b \in Q$ implies $a \oplus b \in Q$), then Q need not be a subalgebra of the D poset $(E; \ominus, 0, 1)$ derived from the given effect algebra (i.e., $a, b \in Q$, $a \leq b$, need not imply $b \ominus a \in Q$).

Definition 1.3. $Q \subseteq P$ is a *sub-D-poset* of a D-poset $(P; \leq, \ominus, 0, 1)$ [a *sub-effect algebra* of an effect algebra $(P; \oplus, 0, 1)$] iff $1 \in Q$ and from elements $a, b, c \in P$ such that $b \ominus a = c$ (equivalently $b = a \oplus c$) at least two are in Q, then $a, b, c \in Q$.

In ref. 14 the compatibility of two elements of a D-poset $(P; \leq, \ominus, 0, 1)$ was introduced. We say that $a, b \in P$ are *compatible* ($a \leftrightarrow b$) if there exist $u, v, w \in P$ such that $a = u \oplus w$ and $b = w \oplus v$, and $u \oplus w \oplus v$ is defined. If P is a lattice, then it is called a *D-lattice* and then $a \leftrightarrow b$ iff $(a \vee b) \ominus b = a \ominus (a \wedge b)$ [14]. In a D-lattice P for every $a, b \in P$ we have $a = (a \wedge b) \oplus (a \ominus (a \wedge b))$ and hence $a \leftrightarrow b$ iff $a \oplus (b \ominus (a \wedge b))$ exists. The notion of compatibility of elements of an effect algebra is defined by the same way.

Lemma 1.4. For elements of a lattice effect algebra $(E; \oplus, 0, 1)$ the following conditions are satisfied:

(i) If $u \leq a$, $v \leq b$, and $a \oplus b$ is defined, then $u \oplus v$ is defined.
(ii) If $b \oplus c$ is defined, then $a \leq b$ iff $a \oplus c \leq b \oplus c$.
(iii) If $a \oplus c$ and $b \oplus c$ are defined, then $(a \oplus c) \vee (b \oplus c) = (a \vee b) \oplus c$.
(iv) $a \leq b$ iff $b' = 1 \ominus b \leq 1 \ominus a = a'$.

The proof is left to the reader.

2. BLOCKS IN *D*-LATTICES AND LATTICE EFFECT ALGEBRAS

Theorem 2.1. Let $(P; \leq, \ominus, 0, 1)$ be a D-lattice and let $x, y, z \in P$ be such that $x \leftrightarrow z$ and $y \leftrightarrow z$. Then

(i) $x \vee y \leftrightarrow z$.
(ii) If $x \leq y$, then $y \ominus x \leftrightarrow z$.
(iii) $x' = 1 \ominus x \leftrightarrow z$.
(iv) $x \wedge y \leftrightarrow z$.
(v) If $x \leq y'$, then $x \oplus y \leftrightarrow z$.

Proof. By assumptions there exist $x \oplus (z \ominus (x \wedge z))$ and $y \oplus (z \ominus (y \wedge z))$.

(i) Since $x \wedge z, y \wedge z \leq (x \vee y) \wedge z \leq z$, we obtain $z \ominus ((x \vee y) \wedge z) \leq z \ominus (x \wedge z), z \ominus (y \wedge z)$, and hence $(x \oplus (z \ominus (x \wedge z))) \vee (y \oplus (z \ominus (y \wedge z))) \geq (x \oplus (z \ominus ((x \vee y) \wedge z))) \vee (y \oplus (z \ominus ((x \vee y) \wedge z))) = (x \vee y) \oplus (z \ominus ((x \vee y) \wedge z))$, which implies that $x \vee y \leftrightarrow z$.

(ii) If $x \leq y$, then $x \wedge z \leq y \wedge z$ and $x \vee z \leq y \vee z$. It follows that there exists $w \in P$ such that $(x \wedge z) \oplus w = y \wedge z$ and $x \vee z = x \oplus (z \ominus (x \wedge z)) \leq y \vee z = y \oplus (z \ominus (y \wedge z)) = (y \wedge z) \oplus (y \ominus (y \wedge z)) \oplus (z \ominus (y \wedge z))$, thus $(x \wedge z) \oplus (x \ominus (x \wedge z)) \oplus (z \ominus (x \wedge z)) \leq (y \wedge z) \oplus (y \ominus (y \wedge z)) \oplus (z \ominus (y \wedge z))$, and since $z = (x \wedge z) \oplus (z \ominus (x \wedge z)) = (y \wedge z) \oplus (z \ominus (y \wedge z))$ we obtain $x \ominus (x \wedge z) \leq y \ominus (y \wedge z)$. The last implies that there is $e \in P$ such that $(x \ominus (x \wedge z)) \oplus e = y \ominus (y \wedge z)$. We obtain $y = (x \wedge z) \oplus w \oplus e \oplus (x \ominus (x \wedge z))$ and $y \oplus (z \ominus (y \wedge z)) = (x \wedge z) \oplus w \oplus e \oplus (x \ominus (x \wedge z)) \oplus (z \ominus (y \wedge z))$. These equalities imply that $y \ominus x = w \oplus e$ and $z = w \oplus [(x \wedge z) \oplus (z \ominus (y \wedge z))]$, since $(x \wedge z) \oplus w = y \wedge z$. We conclude that $y \ominus x \leftrightarrow z$ since $w \oplus e \oplus [(x \wedge z) \oplus (z \ominus (y \wedge z))]$ exists.

(iii) Evidently $1 \leftrightarrow z$ and $x \leq 1$, which implies by (ii) that $x' = 1 \ominus x \leftrightarrow z$.

(iv) By (iii) $x' \leftrightarrow z$ and $y' \leftrightarrow z$, which by (i) implies that $x' \vee y' \leftrightarrow z$ and by (iii) $x \wedge y = (x' \vee y')' \leftrightarrow z$.

(v) $x \oplus y = 1 \ominus (x' \ominus y) \leftrightarrow z$ by conditions (ii) and (iii).

Corollary 2.2. Every maximal subset M of mutually compatible elements of a D-lattice $(P; \leq, \ominus, 0, 1)$ is a sub-D-lattice of P.

Corollary 2.3. Every maximal subset M of mutually compatible elements of a lattice effect algebra $(P; \oplus, 0, 1)$ is a sub-effect-algebra of P.

Definition 2.4. A maximal subset M of mutually compatible elements of a D-lattice (lattice effect algebra) P is called a *block of P*.

Theorem 2.5. Every D-lattice (lattice effect algebra) P is a set-theoretic union of its blocks. Every subset $A \subseteq P$ of mutually compatible elements is contained in a block.

Proof. Let $\emptyset \neq A \subseteq P$ be a set of mutually compatible elements of P and $\mathcal{A} = \{B \subseteq P | A \subseteq B, B$ is a set of mutually compatible elements$\}$. Then every chain $\mathcal{B} \subseteq \mathcal{A}$ (i.e., for $X, Y \in \mathcal{B}$ we have $X \subseteq Y$ or $Y \subseteq X$) the set $\cup\mathcal{B} \in \mathcal{A}$. By the maximal principle there exists a maximal element $M \in \mathcal{A}$. Moreover, for $a \in P$ the set $A = \{0, \mathrm{a}, \mathrm{a}', 1\}$ is mutually compatible.

3. BLOCKS OF *D*-LATTICES (LATTICE EFFECT ALGEBRAS) AND *MV*-ALGEBRAS

The notion of an *MV*-algebra was introduced by Chang [3] for giving an algebraic structure to the infinite-valued Lukasiewicz propositional logics. Later, relations of *MV*-algebras to the theory of linearly ordered groups [4], fuzzy set theory [1], and functional analysis and lattice-ordered groups [15] were shown. Recently Kôpka and Chovanec have shown that *MV*-algebras are Boolean D-posets which are D-lattices of mutually compatible elements [13, 14].

In ref. 15 an *MV*-algebra is defined as follows:

Definition 3.1. An *MV-algebra* is an algebra $(\mathcal{A}, \oplus, *, 0, 1)$, where $\mathcal{A}$ is a nonempty set, 0 and 1 are constant elements of $\mathcal{A}$, $\oplus$ is a binary operation, and $*$ is a unary operation, satisfying the following axioms:

(MVA1) $(a \oplus b) = (b \oplus a)$.
(MVA2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(MVA3) $a \oplus 0 = a$.
(MVA4) $a \oplus 1 = 1$.
(MVA5) $(a^*)^* = a$.
(MVA6) $0^* = 1$.
(MVA7) $a \oplus a^* = 1$.
(MVA8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

The lattice operations $\vee$ and $\wedge$ are defined by the formulas

$$a \vee b = (a^* \oplus b)^* \oplus b \qquad \text{and} \qquad a \wedge b = ((a \oplus b^*)^* \oplus b^*)^*$$

We write $a \leq b$ iff $a \vee b = b$. The relation $\leq$ is a partial ordering on $\mathcal{A}$, and $0 \leq a \leq 1$ for any $a \in \mathcal{A}$. An *MV*-algebra is a distributive lattice with respect to the operations $\vee$ and $\wedge$. We put

$$b \ominus a := (a \oplus b^*)^* \qquad \text{for} \quad a \leq b, \quad a, b \in P$$

Then an *MV*-algebra $\mathscr{A}$ becomes a *D*-lattice, more precisely, a distributive *D*-lattice [14].

Conversely, if $(M, \leq, \ominus, 0, 1)$ is a *D*-lattice of mutually compatible elements and we put $a^* = 1 \ominus a$, $a - b = a \ominus (a \wedge b)$, and $a \oplus b = (a^* - b)^*$ for all $a, b \in M$, then $(M; \oplus, *, 0, 1)$ is and *MV*-algebra [14].

Theorem 3.2. Every *D*-lattice (lattice effect algebra) P is a set-theoretic union of *MV*-algebras being blocks of P.

Remark 3.3. It is known that if $x \vee x' = 1$ for every element x of a *D*-lattice $(P; \leq, \ominus, 0, 1)$, then $(P; \leq, ', 0, 1)$ is an orthomodular lattice [13, 9]. In such a case blocks of a *D*-lattice P become maximal Boolean subalgebras of orthomodular lattice P [9, 16]. This is because an *MV*-algebra M in which $x \vee x' = 1$ for all $x \in M$ is a Boolean algebra.

The notion of a central element of an effect algebra $(E; \oplus, 0, 1)$ was introduced by Greechie *et al.* [7]:

Definition 3.3. For an effect algebra $(E; \oplus, 0, 1)$ an element $z \in E$ is called *central* iff for every $x \in E$ there exist $x \wedge z$ and $x \wedge z'$ and $x = (x \wedge z) \vee (x \wedge z')$. The set $C(E)$ of all central elements of E is called the *center of E*.

Remark 3.4. In ref. 17 it has been shown that for a lattice effect algebra $(E; \oplus, 0, 1)$ an element $z \in E$ is central iff $z \wedge z' = 0$ and $z \leftrightarrow x$ for all $x \in E$. In ref. 5 an element $z \in E$ is called isotropic iff $z \oplus z$ is defined in E. We obtain the following statement:

Theorem 3.5. The intersection $M(E)$ of all blocks of a lattice effect algebra $(E; \oplus, 0, 1)$ is the center of E iff no element of $M(E)$ is isotropic.

4. BLOCKS OF *D*-LATTICES (LATTICE EFFECT ALGEBRAS) ARE CLOSED SETS IN ORDER TOPOLOGY

Recall that for a net $(x_\alpha)_{\alpha \in \varepsilon}$ of elements of a poset $(P; \leq)$ (i.e., a subset of P indexed by a directed set ε) we say that x_α *order converges* to $x \in P$ iff there exist nets $(u_\alpha)_{\alpha \in \varepsilon}$, $(v_\alpha)_{\alpha \in \varepsilon}$ such that $u_\alpha \leq x_\alpha \leq v_\alpha$ for all α and $u_\alpha \uparrow x$, $v_\alpha \downarrow x$. Here $x_\alpha \uparrow x$ means that $\vee\{x_\alpha | \alpha \in \varepsilon\} = x$ and $u_{\alpha_1} \leq u_{\alpha_2}$ for all $\alpha_1 \leq \alpha_2$. The symbol $v_\alpha \downarrow x$ is dual.

Lemma 4.1. Let $(P; \leq, \ominus, 0, 1)$ be a *D*-poset. Let $(u_\alpha)_{\alpha \in \varepsilon}, (v_\alpha)_{\alpha \in \varepsilon} \subseteq P$ be nets such that $u_\alpha \leq v_\alpha$ for all α and $u_\alpha \uparrow x$, $v_\alpha \downarrow y$. Then $x \leq y$ and $v_\alpha \ominus u_\alpha \downarrow y \ominus x$.

Proof. For every $\alpha, \beta \in \varepsilon$ there exists $\gamma \in \varepsilon$ such that $\alpha, \beta \leq \gamma$ and hence $u_\alpha \leq u_\gamma \leq v_\gamma \leq v_\beta \downarrow y$. It follows that $u_\alpha \leq y$ for all $\alpha \in \varepsilon$ and hence $x \leq y$. Moreover, $u_\alpha \leq x \leq y \leq v_\alpha \Rightarrow y \ominus x \leq v_\alpha \ominus u_\alpha$. If $z \leq v_\alpha \ominus u_\alpha$ for all α, then $z \oplus u_\alpha \uparrow z \oplus x$ and $z \oplus u_\alpha \leq v_\alpha \downarrow y$. The last implies that $z \oplus x \leq y$, which implies that $z \leq y \ominus x$. We obtain that $y \ominus x = \wedge\{v_\alpha \ominus u_\alpha | \alpha \in \varepsilon\}$.

Lemma 4.2. Let $(x_\alpha)_{\alpha \in \varepsilon} \subseteq P$ be a net of elements of a D-lattice $(P; \leq, \ominus, 0, 1)$. If $x_\alpha \xrightarrow{(o)} x$ and $x_\alpha \leftrightarrow y$ for all α, then $x \leftrightarrow y$.

Proof. Since $x_\alpha \xrightarrow{(o)} x$, there are nets $(u_\alpha)_{\alpha \in \varepsilon}, (v_\alpha)_{\alpha \in \varepsilon} \subseteq P$ such that $u_\alpha \uparrow x$, $v_\alpha \downarrow y$, and $u_\alpha \leq x_\alpha \leq v_\alpha$ for all $\alpha \in \varepsilon$. The assumption $x_\alpha \leftrightarrow y$ implies that $x_\alpha \vee y = x_\alpha \oplus (y \ominus (x_\alpha \wedge y)) \geq u_\alpha \oplus (y \ominus (v_\alpha \wedge y))$. Moreover, $x_\alpha \wedge y \leq v_\alpha \wedge y \leq y$, which implies that $y \ominus (v_\alpha \wedge y) \leq y \ominus (x_\alpha \wedge y)$. Evidently, $v_\alpha \wedge y \downarrow x \wedge y$ and $y \ominus (v_\alpha \wedge y) \uparrow y \ominus (x \wedge y)$. Further, $y \ominus (v_\alpha \wedge y) \leq 1 \ominus u_\alpha \downarrow 1 \ominus x$ and hence $y \ominus (x \wedge y) \leq 1 \ominus x$. It follows that $x \oplus (y \ominus (x \wedge y))$ is defined. We conclude that $x \leftrightarrow y$.

Recall that a subset F of a poset $(P; \leq)$ is a closed set in the order topology τ_o on P iff F contains all order limits of order-convergent nets of elements of F. Thus Lemma 4.2 has the following corollary:

Theorem 4.3. Every block of a D-lattice $(P; \leq, \ominus, 0, 1)$ [lattice effect algebra $(P; \oplus, 0, 1)$] is a τ_o-closed set in order topology τ_o on P.

Corollary 4.4. Let $M \subseteq P$ be a block of a D-lattice $(P; \leq, \ominus, 0, 1)$ [lattice effect algebra $(P; \oplus, 0, 1)$] and $A \subseteq M$. Then:

(i) If $\vee A$ exists in P, then $\vee A \in M$
(ii) If $\wedge A$ exists in P, then $\wedge A \in M$.

Proof. Assume that $\varepsilon = \{\alpha \subseteq A | \alpha \text{ is finite}\}$ and put $u_\alpha = \vee\alpha$, $v_\alpha = \wedge\alpha$ for every $\alpha \in \varepsilon$. If $\vee A$ exists, then $u_\alpha \uparrow \vee A$ and by Theorem 4.3 we obtain $\vee A \in M$. If $\wedge A$ exists, then $v_\alpha \downarrow \wedge A$ and $\wedge A \in M$ by Theorem 4.3.

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